# MODEL OF THE TURBULENT DIFFUSION OF AEROSOLS

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A model of turbulent diffusion of aerosols in the atmosphere is proposed, on whose basis a Lagrange correlation function of the cloud particle velocity is constructed.

Aerosol propagation in the atmosphere has been examined repeatedly in the literature [1-3]. One of the best known theories is that of Sutton [1]. Let us recall that in the case of homogeneous turbulence, the root-mean-square displacement of aerosol cloud particles along the y axis up to the time t, say, can be expressed in terms of the normalized Lagrange correlation function of the particle velocity $R_L(\xi)$ [4]:

$$\sigma_y^2(t) = 2\,\overline{(v_y')^2} \int_0^t dt' \int_0^{t'} R_L(\xi)\, d\xi. \tag{1}$$

Here $\overline{(v_y')^2}$ is the y-component of the particle velocity fluctuations, and the vector of the mean wind velocity is directed along the x axis. Sutton proposed use of the following function

$$R_L(\xi) = \left(1 + \frac{\xi}{\tau}\right)^{-n}, \tag{2}$$

where $0 < n < 1$, $\tau = \nu/\overline{(u_y')^2}$, $\nu$ is the molecular viscosity. There is then obtained for $\sigma_y^2(t)$

$$\sigma_y^2(t) = \frac{1}{2} C_y^2 (\bar{u}t)^{2-n}, \tag{3}$$

where

$$C_y^2 = \frac{4\nu^n}{(1-n)(2-n)\bar{u}^n} \left(\frac{\overline{(u_y')^2}}{\bar{u}^2}\right)^{1-n} \tag{4}$$

is the generalized turbulent diffusion coefficient. In later papers, Sutton introduced the concept of macroviscosity to replace the molecular viscosity. Analogous expressions can also be written for $\sigma_z^2$, $C_z^2$. The parameter n is determined empirically. Moreover, Sutton uses other values of $C_y$ obtained experimentally and not from (4), in the examples cited, where the experimental $C_y$ diminishes with altitude.

A discussion of the advantages and disadvantages of the Sutton formulas in the literature [5, 6] indicates that they are substantially empirical and without theoretical foundation, although the Sutton model was used extensively in practice, particularly in England and the USA.

Let us show that the model we propose for propagation of an aerosol cloud in the atmosphere affords the possibility of giving a foundation to that correlation function $R_L(\xi)$ which can take on the Sutton form under specific conditions, while the parameters n and $C_y$ are interpreted within the framework of our model.

Let us consider the behavior of a dense aerosol cloud in a turbulent atmosphere, whose dynamic state is characterized by the turbulent viscosity coefficient K. In the case of a high particle concentration in the cloud, the mutual interaction forces are so substantial that the cloud initially behaves as a single whole moving at the mean velocity of the wind. Let us assume that the cloud starts to be broken up into separate "fragments," puffs, under the influence of atmospheric turbulence and that the radial distribution of such puffs (we assume them spherical for simplicity) is log-normal. Kolmogorov [7] proved the applicability of this distribution to random breakdown processes.

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Each puff moves as a whole, subject to further breakdown as it moves. The cloud center of mass continues to move at the mean wind velocity $\bar{u}$. Let $R_0$ denote the greatest of the probable dimensions of the puffs being formed. This quantity should evidently be related to some actual characteristics of the state of the atmosphere. For example, as we have conceived, it should correspond to the dimensions of the greatest of the atmospheric vortices taking part in destruction of the cloud. Each particle moves at the velocity of the puff to which it belongs at the given time. The density $\rho_p$ of all the puffs is constant. The log-normal distribution is written as [7]

$$w(x, t)\,dx = \frac{1}{\sqrt{2\pi t}B} e^{-\frac{(x-At)^2}{2B^2 t}}\,dx, \tag{5}$$

where $x = \ln R/R_0$, $At = \bar{x}$, $B^2 t = \overline{(x-\bar{x})^2}$, $R < R_0$, $A < 0$. Let us assume that the atmospheric vortices jolt the puffs which have been formed (and are continuously being formed) exactly as molecules jolt Brownian particles and just as a Brownian particle is in thermal equilibrium with the molecules of the medium, each puff is in equilibrium with the atmospheric vortices, i.e., $\overline{(v')^2} = \overline{(u')^2}$.

Let us write the equation of motion of a puff of radius R and mass m along the y axis as follows:

$$m\frac{dv}{dt} = -\alpha v(t) + f(t), \tag{6}$$

where v is the velocity of a fragment, f(t) is a random force acting on the puff from the atmospheric vortices, and $\alpha = 6\pi R K \rho_{air}$. Since $m = (4/3)\pi R^3 \rho_p$ ($\rho_{air}$ is the air density), then

$$\frac{dv}{dt} = -a(R)v(t) + b(R, t), \tag{7}$$

where

$$a(R) = \frac{9}{2}K\frac{\rho_{air}}{\rho_p}\frac{1}{R^2}, \qquad b(R, t) = \frac{3}{4\pi\rho_p}\frac{1}{R^3}f(t). \tag{8}$$

It is necessary to track some isolated cloud particle belonging to fragments of smaller and smaller size at successive times in conformity with the mechanism of cloud fragmentation. The equation of motion of such a particle is therefore described also by (7), but with radius R dependent on time in a random way. For simplicity, let us assume that the parameters A and B of the distribution (5) are such that

$$B^2 \ll A^2 t. \tag{9}$$

Then the equation of motion of any cloud particle is written as

$$\frac{dv}{dt} = -\bar{a}(R, t)v(t) + \bar{b}(R, t), \tag{10}$$

where

$$\bar{a}(R, t) = \frac{9}{2}\frac{\rho_{air}}{\rho_p}\frac{K}{R_0^2}e^{-2At}, \tag{11}$$

$$\bar{b}(R, t) = \frac{3}{4\pi\rho_p}f(t)\frac{1}{R_0^3}e^{-3At}. \tag{12}$$

Using the notation $-2A \equiv k$, $(9/2)K(\rho_{air}/\rho_p)1/R_0^2 \equiv p$ and noting that $(3/4\pi\rho_p)(1/R_0^3)e^{-3At} = 1/\bar{m}^R$, let us write (10) with the initial condition $v(0) = 0$:

$$v(t) = e^{-\frac{p}{k}e^{kt}} \int_0^t \frac{f(s)}{\bar{m}^R(s)} e^{\frac{p}{k}e^{ks}}\,ds. \tag{13}$$

Let us find the correlation function $\overline{v'(t)v'(t')}$ where the bar denotes the time average:

$$\overline{v'(t)\,v'(t')} = e^{-\frac{p}{k}[e^{kt}+e^{kt'}]} \int_0^t\int_0^{t'} \frac{\overline{f(s)f(s')}}{\bar{m}^R(s)\,\bar{m}^R(s')} e^{\frac{p}{k}[e^{ks'}+e^{ks}]}\,ds'ds. \tag{14}$$

Most important here is the correct selection of the correlation function of the random forces $\overline{f(s)f(s')}$. In the theory of Brownian motion $\overline{f(s)f(s')} = 2\xi m\overline{(v')^2}\delta(s-s')$, where $\xi$ is the friction coefficient and m is the mass of the Brownian particle. Let us construct our $\overline{f(s)f(s')}$ by analogy with the Brownian motion. It is seen from (10) that the role of the friction coefficient in our case is played by $\bar{a}^R(t)\bar{m}^R(t)$, where the bar with subscript R denotes averaging with respect to R. Then

$$\overline{v'(t)\,v'(t')} = \overline{(v')^2}\, e^{-\frac{p}{k}(e^{kt}+e^{kt'})}\,[e^{\frac{2p}{k}e^{kt'}} - e^{\frac{2p}{k}}]. \tag{15}$$

Let $e^{kt'} \equiv n$. The second number in the square brackets in (15) can be neglected if

$$e^{\frac{2p}{k}(n-1)} \gg 1. \tag{16}$$

Therefore

$$\overline{v'(t)\,v'(t')} = \overline{(v')^2}\, e^{-\frac{p}{k}(e^{kt}-e^{kt'})}. \tag{17}$$

Using the notation $t-t' \equiv \xi$ and dividing by $\overline{(v')^2}$, we obtain

$$R_L(\xi) = \frac{\overline{v'(t)\,v'(t')}}{\overline{(v')^2}} = e^{-\frac{p}{k}(e^{k(\xi+t')}-e^{kt'})}. \tag{18}$$

Hence, for all $\xi \ll 1/k$ and $\xi \ll 1/p$

$$R_L(\xi) = \left(1+\frac{\xi}{\tau}\right)^{-n}, \tag{19}$$

where $\tau = 1/p$. If the time domain near $t' \ll 1/k$ is integrated, then n is a number not much greater than one which varies slightly with time. Taking into account that $1/p$ equals $(2/9)(R_0^2/K)\rho_p/\rho_{air}$ and is the order of magnitude of the Lagrange scale of turbulence, as also taking account of (16), the deduction can be made that the Sutton correlation function obtained is valid for times less than the Lagrange scale of turbulence.

Let us compute the root-mean-square displacement of a cloud particle by means of (1) by using the "exact" correlation function (18) and the equality $\overline{(v')^2} = \overline{(u')^2}$:

$$\sigma_y^2(t) = \overline{(u'_y)^2}\cdot\frac{2^{2-n}}{3-n}\,\tau^{n-1}t^{3-n}. \tag{20}$$

According to (3) the generalized coefficient of turbulent diffusion $C_y$ is determined from the relationship

$$\sigma_y^2(t) = \frac{C_y^2}{2}(\bar{u}t)^{\gamma}, \tag{21}$$

where $\gamma = 3-n$ as follows from (20). Therefore, $C_y$ will equal

$$C_y = \sqrt{\frac{\overline{(u'_y)^2}}{\bar{u}^{3-n}}\,\frac{2^{3-n}}{3-n}\,\tau^{n-1}} = \sqrt{\frac{\overline{(u'_y)^2}}{\bar{u}^{3-n}}\,\frac{4}{3-n}\left(\frac{R_0^2}{9K}\,\frac{\rho_p}{\rho_{air}}\right)^{n-1}}. \tag{22}$$

A computation of $C_y$ by means of (22) by using experimental values for $R_0$ and K from [8] shows that $C_y$ diminishes with altitude for the altitude range to 250 m.

## NOTATION

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| $R_L(\xi)$ | is the Lagrange correlation function of the particle velocity; |
| v | is the particle velocity; |
| v' | is the fluctuation in the particle velocity; |
| $\sigma^2$ | is the root-mean-square particle displacement; |
| $\nu$ | is the molecular viscosity of the medium; |
| K | is the turbulent viscosity; |
| $C_y$ | is the generalized coefficient of turbulent diffusion; |
| A, B | are the parameters of the log-normal distribution; |
| R | is the fragment radius; |
| $R_0$ | is the dimension of the atmospheric vortices; |
| $\bar{u}$ | is the mean wind velocity; |
| u' | is the fluctuation in the wind velocity. |

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